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COMMENT

Generalised KS transformation: from five-dimensional hydrogen atom to eight-dimensional isotrope oscillator

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Abstract. It is shown that non-bijective quadratic transformation generated by the Kelly matrix changes the problem of a five-dimensional hydrogen atom into the problem of an eight-dimensional isotrope oscillator.

It is known (Kibler and Négadi 1984) that the problem of a hydrogen atom can be reduced to the problem of a four-dimensional isotrope oscillator by means of the so-called KS transformation. Does this mean that many well developed methods applied in quantum field theory and nuclear physics can also be effectively used in investigations of the behaviour of hydrogen-like atoms in external electric and magnetic fields? What are the relations between dynamic symmetry groups of the hydrogen atom and the four-dimensional isotrope oscillator? What is the explicit form of the transformations binding hydrogen atom wavefunctions and four-oscillator wavefunctions? Can the above-mentioned phenomenon be generalised to other dimensionalities, i.e. can an f-dimensional hydrogen atom be reduced to an isotrope oscillator of any dimensionality? The first question is treated optimistically by Kibler and Négadi (1984) and the second was studied by Chen (1982). The third question was formulated and developed for computation by Kibler *et al* (1986) and was then accurately, or analytically, solved by Mardoyan *et al* (1986). We comment now on the fourth question. If $f=2$, transition from the hydrogen atom to the oscillator takes place through the transformation $x=u_1^2-u_2^2$, $y=2u_1u_2$. There is no need for greater space dimensionality, such as $f=3$. Like KS transformation, this example belongs to the class of so-called non-bijective quadratic transformations whose most important feature is the validity of the Euler identity

$$(u_1^2+\ldots+u_n^2)=x_1^2+\ldots+x_f^2. \tag{1}$$

In our case x_i and u_k are the hydrogen atom and isotrope oscillator coordinates, respectively. It is known (Zhevlakov *et al* 1978) that the identity (1) is only valid for the following four pairs, made of dimensionality indices of spaces n and f:

$$(f, n)=(1,1);(2,2);(3,4);(5,8). \tag{2}$$

The above-mentioned transformations correspond to the pairs (2, 2) and (3, 4). A question arises as to whether the transformation related to the pair (5, 8) changes the problem of a five-dimensional hydrogen atom into the problem of an eight-dimensional isotrope oscillator. This paper positively answers that question. Thus, except for the pair (1, 1), the set (2) covers all possible non-bijective quadratic transformations changing the hydrogen problem into the oscillator one.

Let us introduce a non-bijective quadratic transformation, binding coordinates $(x_1, \ldots, x_5)$ of the space R^5 with coordinates $(u_1, \ldots, u_8)$ of the space R^8,

$$q_m = \sum_{j=1}^{8} T_{mj} u_j \tag{3}$$

where $\boldsymbol{q} = (x_1, \ldots, x_5, 0, 0, 0)$, and T_{mj} has the form

$$T_{mj} = \begin{bmatrix} u_2 & u_1 & u_4 & u_3 & u_6 & u_5 & -u_8 & -u_7 \\ u_4 & -u_3 & -u_2 & u_1 & u_8 & u_7 & u_6 & u_5 \\ u_6 & -u_5 & -u_8 & -u_7 & -u_2 & u_1 & -u_4 & -u_3 \\ u_8 & u_7 & u_6 & -u_5 & -u_4 & u_3 & u_2 & u_1 \\ u_1 & -u_2 & u_3 & -u_4 & u_5 & u_6 & u_7 & -u_8 \\ u_3 & u_4 & -u_1 & -u_2 & u_7 & -u_8 & -u_5 & u_6 \\ u_5 & u_6 & -u_7 & u_8 & -u_1 & -u_2 & u_3 & -u_4 \\ u_7 & -u_8 & u_5 & u_6 & -u_3 & -u_4 & u_1 & u_2 \end{bmatrix}. \tag{4}$$

The matrix (4) differs from the usual Kelly matrix (Zhevlakov *et al* 1978, Kibler and Lambert 1986) by a certain rearrangement of lines. Expressions (3) and (4) yield bilinear relations for coordinates $\boldsymbol{x}$ and $\boldsymbol{u}$:

$$\begin{aligned} x_1 &= 2(u_1u_2 + u_3u_4 + u_5u_6 - u_7u_8) \\ x_2 &= 2(u_1u_4 - u_2u_3 + u_5u_8 + u_6u_7) \\ x_3 &= 2(u_1u_6 - u_2u_5 - u_3u_8 - u_4u_7) \\ x_4 &= 2(u_1u_8 + u_2u_7 + u_3u_6 - u_4u_5) \\ x_5 &= u_1^2 + u_3^2 + u_5^2 + u_7^2 - u_2^2 - u_4^2 + u_6^2 - u_8^2. \end{aligned} \tag{5}$$

It should be stressed that non-bijectivity of the $R^5 \leftarrow R^8$ transformation means that for each element of R^5 there is a whole set of elements in R^8; this set is called a fibre.

Using an explicit form of matrix (4) one can easily show that

$$\sum_{j=1}^{8} T_{mj} T_{nj} = u^2 \delta_{mn} \tag{6}$$

$$\sum_{j=1}^{8} \frac{\partial T_{mj}}{\partial u_j} = 0 \tag{7}$$

where $u^2 \equiv u_1^2 + \ldots + u_8^2$. The Euler identity (1) has the form

$$x^2 \equiv x_1^2 + \ldots + x_5^2 = u^4 \tag{8}$$

and can be checked by formulae (3) and (6), or by (5). It also follows from (4) that

$$\frac{\partial x_m}{\partial u_j} = 2T_{mj}. \tag{9}$$

Here the index m has values $1, \ldots, 5$.

Now we switch to a transformation of derivatives. Taking into account (9), we obtain

$$\frac{\partial \mathcal{F}(u)}{\partial u_j} = 2 \sum_{k=1}^{5} T_{kj} \frac{\partial \mathcal{F}(u)}{\partial x_k} \tag{10}$$

where $\mathscr{F}(u)$ is an arbitrary function of $\boldsymbol{u}$. Multiplying (10) by T_{mj}, summing over j and bearing in mind the condition (6), one can easily show that

$$\frac{\partial \mathscr{F}(u)}{\partial x_m} = \frac{1}{2u^2} \sum_{j=1}^{8} T_{mj} \frac{\partial \mathscr{F}(u)}{2u_j} \tag{11}$$

(remember that $m = 1, \ldots, 5$ in (11)). We introduce the notation

$$\frac{\partial}{\partial q_m} \equiv \frac{1}{2u^2} \sum_{j=1}^{8} T_{mj} \frac{\partial}{\partial u_j} \tag{12}$$

where $m = 1, \ldots, 8$. Formula (12) generalises relation (11) to the case with $m = 6, 7, 8$.

Applying an inverse transformation to (11) and using (6), we obtain an operator formula which generalises (10):

$$\frac{\partial}{\partial u_j} = 2 \sum_{m=1}^{8} T_{mj} \frac{\partial}{\partial q_m}. \tag{13}$$

Let us consider the second derivatives. It follows from (12) that

$$\sum_{m=1}^{8} \frac{\partial^2 \mathscr{F}(u)}{\partial q_m^2} = \tfrac{1}{2} \sum_{m=1}^{8} \sum_{j=1}^{8} \frac{\partial}{\partial q_m} \left(\frac{1}{u^2} T_{mj} \frac{\partial \mathscr{F}(u)}{\partial u_j} \right). \tag{14}$$

Applying formula (12) once more with allowance for (6), we obtain

$$\sum_{m=1}^{8} \frac{\partial^2 \mathscr{F}(u)}{\partial q_m^2} = \frac{1}{4u^2} \sum_{m=1}^{8} \sum_{j=1}^{8} \sum_{\nu=1}^{8} T_{m\nu} \left(\frac{\partial}{\partial u_\nu} \frac{1}{u^2} T_{mj} \right) \frac{\partial \mathscr{F}(u)}{\partial u_j} + \frac{1}{4u^2} \sum_{\nu=1}^{8} \frac{\partial^2 \mathscr{F}(u)}{\partial u_\nu^2}.$$

Differentiating by parts in the first term, employing condition (6) and then identity (7), the result is

$$\sum_{m=1}^{8} \frac{\partial^2 \mathscr{F}(u)}{\partial q_m^2} = \frac{1}{4u^2} \sum_{\nu=1}^{8} \frac{\partial^2 \mathscr{F}(u)}{\partial u_\nu^2}. \tag{15}$$

Relation (15) and identity (8) are the only mathematical tools required for the transformation of the Hamiltonian of the five-dimensional hydrogen atom into the Hamiltonian of the eight-dimensional isotrope oscillator. Before getting down to this transformation, let us return to formula (12) and introduce operators

$$\hat{\mathscr{L}}_m = \mathrm{i} u^2 \frac{\partial}{\partial q_m} = \tfrac{1}{2}\mathrm{i} \sum_{j=1}^{8} T_{mj} \frac{\partial}{\partial u_j} \qquad m = 6, 7, 8. \tag{16}$$

It follows from this definition and formula (9) that

$$\hat{\mathscr{L}}_m x_k = \mathrm{i} u^2 \delta_{mk} = 0$$

since $m = 6, 7, 8$, and $k = 1, \ldots, 5$. Thus, operators (16) do not depend upon coordinates $x_1, \ldots, x_5$ and the following identity is valid for an arbitrary function $F(\boldsymbol{x})$:

$$\hat{\mathscr{L}}_m F(\boldsymbol{x}) = 0. \tag{17}$$

Using formula (16) and matrix (4), we can prove that operators (16) have the following explicit form:

$$\hat{\mathscr{L}}_6 = \frac{\mathrm{i}}{2} \left(u_3 \frac{\partial}{\partial u_1} + u_4 \frac{\partial}{\partial u_2} - u_1 \frac{\partial}{\partial u_3} - u_2 \frac{\partial}{\partial u_4} + u_7 \frac{\partial}{\partial u_5} - u_8 \frac{\partial}{\partial u_6} - u_5 \frac{\partial}{\partial u_7} + u_6 \frac{\partial}{\partial u_8} \right)$$

$$\hat{\mathscr{L}}_7 = \frac{\mathrm{i}}{2} \left(u_5 \frac{\partial}{\partial u_1} + u_6 \frac{\partial}{\partial u_2} - u_7 \frac{\partial}{\partial u_3} + u_8 \frac{\partial}{\partial u_4} - u_1 \frac{\partial}{\partial u_5} - u_2 \frac{\partial}{\partial u_6} + u_3 \frac{\partial}{\partial u_7} - u_4 \frac{\partial}{\partial u_8} \right)$$

$$\hat{\mathscr{L}}_8 = \frac{\mathrm{i}}{2} \left(u_7 \frac{\partial}{\partial u_1} - u_8 \frac{\partial}{\partial u_2} + u_5 \frac{\partial}{\partial u_3} + u_6 \frac{\partial}{\partial u_4} - u_3 \frac{\partial}{\partial u_5} - u_4 \frac{\partial}{\partial u_6} - u_1 \frac{\partial}{\partial u_7} + u_2 \frac{\partial}{\partial u_8} \right).$$

Now let us combine the problem of the five-dimensional hydrogen atom with that of the eight-dimensional isotrope oscillator. Let $(x_1, \ldots, x_5)$ and $(u_1, \ldots, u_8)$ be cartesian coordinates of the two problems respectively. Using formula (15) with allowance for (16) one obtains the following relation between Laplacians Δ_x and Δ_u:

$$\Delta_x = \frac{1}{4u^2}\Delta_u + \frac{1}{u^4}\hat{\mathscr{L}}^2 \tag{18}$$

where the operator $\hat{\mathscr{L}}^2$ is defined as

$$\hat{\mathscr{L}}^2 = \hat{\mathscr{L}}_6^2 + \hat{\mathscr{L}}_7^2 + \hat{\mathscr{L}}_8^2. \tag{19}$$

Relation (18) and formulae (17) and (19) allow transformation of the Schrödinger equation for the five-dimensional hydrogen atom

$$\left(-\frac{\hbar^2}{2\mu}\Delta_x - \frac{e^2}{r}\right)\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x}) \tag{20}$$

into the form

$$\left(-\frac{\hbar^2}{2\mu}\Delta_u - 4Eu^2\right)\psi(\boldsymbol{x}) = 4e^2\psi(\boldsymbol{x}). \tag{21}$$

Compare the obtained equation with the Schrödinger equation for the eight-dimensional isotrope oscillator

$$\left(-\frac{\hbar^2}{2\mu}\Delta_u + \frac{\mu\omega^2u^2}{2}\right)\phi(\boldsymbol{u}) = \varepsilon\phi(\boldsymbol{u}). \tag{22}$$

Equation (22) can be solved from the point of view of physics on condition that

$$\varepsilon/\hbar\omega - 4 = N \qquad N = 0, 1, 2, \ldots. \tag{23}$$

It follows from (23) that at a given ω then ε is quantised (this is a standard situation) and vice versa—at a given ε, the frequency ω is quantised. The function $\psi(\boldsymbol{x})$ will be a partial solution of equation (22) satisfying the identity

$$\hat{\mathscr{L}}_m\phi(\boldsymbol{u}) = 0$$

on condition that

$$\tfrac{1}{2}\mu\omega^2 = -4E \qquad 4e^2 = \varepsilon. \tag{24}$$

Besides, it follows from (5) that $\psi(\boldsymbol{x})$ is an even function of variables u:

$$\psi(\boldsymbol{x}(\boldsymbol{u})) = \psi(\boldsymbol{x}(-\boldsymbol{u})).$$

That is why for any solution of equation (20), $\psi(\boldsymbol{x})$ can be expanded in a full system of even solutions $\phi_{N\alpha}$ (α denotes all other quantum numbers) of equation (22), i.e.

$$\psi_n(\boldsymbol{x}) = \sum_\alpha C_{n\alpha}\phi_{N\alpha}(\boldsymbol{u})$$

where

$$N = 2n. \tag{25}$$

It is easy to see that n coincides with the main quantum number of the five-dimensional hydrogen atom. Substituting the second condition from (24) and (25) into (23), we get

$$\omega_n = 2e^2/\hbar(n+2). \tag{26}$$

Thus in our case ε is given and ω is quantised. Now substituting (26) into the first condition of (24), we obtain an expression defining the energy spectrum of the five-dimensional hydrogen atom

$$E_n = -\frac{\mu e^4}{2\hbar^2}\frac{1}{(n+2)^2}.$$

Let us return to operators (16) and denote them in another way

$$\hat{\mathscr{J}}_1 = \hat{\mathscr{L}}_6 \qquad \hat{\mathscr{J}}_2 = \hat{\mathscr{L}}_7 \qquad \hat{\mathscr{J}}_3 = \hat{\mathscr{L}}_8. \tag{27}$$

Employing the explicit form of operators one can prove by direct calculation that operators (27) satisfy commutation relations

$$\hat{\mathscr{J}}_i\hat{\mathscr{J}}_k - \hat{\mathscr{J}}_k\hat{\mathscr{J}}_i = \mathrm{i}e_{ikm}\hat{\mathscr{J}}_m$$

where $i, k = 1, 2, 3$. Calculations show that operators (27) commute with the oscillator Hamiltonian

$$\hat{\mathscr{H}}_0 = -\frac{\hbar^2}{2\mu}\Delta_u + \frac{\mu\omega^2 u^2}{2} \qquad \hat{\mathscr{J}}_i\hat{\mathscr{H}}_0 - \hat{\mathscr{H}}_0\hat{\mathscr{J}}_i = 0.$$

Thus there are states where the energy of the oscillator, the projection of the 'moment' $\mathscr{J}_z$ and the squared moment $\mathscr{J}(\mathscr{J}+1)$ can be measured at the same time.

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